Return Models - Part III
A Continuous-Time Model With Mean Reversion

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We will define the variables m and n to be time in years where $0 \le m < n \le \infty$. In this white paper we will build a continuous-time, mean-reverting valuation model to calculate the discounted value at time m of expected cash flow to be received over the time interval $[m, n]$ from the perspective of time zero (i.e. the present). The valuation integral takes the following form...

$$\text{Value at time } m = \int\limits_m^n \text{Annualized net cash flow at time } t \times \text{Discount factor at time } t \times \delta t \tag{1}$$

We will define enterprise value to be the discounted value of after-tax operating net cash flow before debt service and debt tax shield value to be the discounted value of tax savings applicable to the tax-deductibility of interest payments on debt. We will define company value to be the sum of enterprise value and debt tax shield value. This statement in equation form is...

$$\text{Company value} = \text{Enterprise value} + \text{Debt tax shield value} \tag{2}$$

We will define the variable N_t to be notional value at time t. Notional value can be defined as annualized revenue, tangible assets, annualized production, annualized unit sales, etc. We will define the variable B_t to be base value at time t. Base value can be defined as tangible assets (non-financial companies) or tangible capital (financial companies). The equation that defines the relationship between notional and base value is...

$$B_t = \phi\, N_t \text{ ...where... } \phi = \frac{B_0}{N_0} \tag{3}$$

In the discrete-time model from Part I the notional value growth rate and the base value rate of return were constants, which may prove to be problematic. If the current notional value growth rate and/or base value rate of return are not sustainable in the long-run then those short-term unsustainable rates should revert to the long-term sustainable rates over time (i.e. mean reversion).

$$\text{Rate at time } t = \text{Long-term rate} + (\text{Short-term rate} - \text{Long-term rate}) \times \text{Rate of decay} \tag{4}$$

Note that the decay rate in Equation (4) above is a value greater than zero and less than one. To assist us in building our model we will work through the following hypothetical problem from Part I with added variables to allow for mean reversion...

Our Hypothetical Problem

ABC Company is a company in the retail industry. ABC Company has a revenue growth rate and return on assets that are unsustainable in the long-term and will increase/(decrease) to the sustainable rates over time. For modeling purposes we will define notional value and base value to be the following...

Table 1: Definitions

Notional value	Annualized operating revenue
Base value - Enterprise value	Tangible operating assets
Base value - Debt tax shield value	Debt principal balance

The table below presents ABC Company's go-forward model assumptions...

Table 2: Model Assumptions

Description	Balance
Notional value (annualized revenue) at time zero ($ in thousands)	$1,000,000
Notional value short-term unsustainable growth rate (RGR) (%)	10.00
Notional value long-term sustainable growth rate (RGR) (%)	4.00
Base value (tangible assets) at time zero ($ in thousands)	$1,250,000
Base value short-term unsustainable rate of return (ROA) (%)	18.00
Base value long-term sustainable rate of return (ROA) (%)	12.00
Debt principal balance at time zero ($ in thousands)	$300,000
Debt interest rate (%)	6.00
Income tax rate (%)	20.00
Discount rate (%)	12.00
Transition half-life in years (#)	3.00

We are tasked with answering the following questions:

Question 1: What is company value at time zero given that cash flow is expected to be received over the time interval $[0,\infty]$?

Question 2: What is company value at the end of 5 five given that cash flow is expected to cease after year 15?

We will define the variable κ to be the continuous-time cost of capital. Using the data in Table 2 above the equation for the cost of capital is...

$$\kappa = \ln(1+\text{discrete-time annual rate}) = \ln(1+0.1200) = 0.1133 \tag{5}$$

Mean Reversion

We will define the rates at time zero to be the unsustainable rates and the rates at time infinity to be the sustainable rates. We will define the variable λ to be the rate of mean reversion, which is the rate at which that the unsustainable rates transition to the sustainable rates over time. The equation for the expected rate at time t from the perspective of time zero is...

$$\text{Rate at time } t = \text{Rate at time infinity} + \Bigg(\text{Rate at time zero} - \text{Rate at time infinity}\Bigg)\,\text{Exp}\Bigg\{-\lambda\,t\Bigg\} \tag{6}$$

Note that the limit of Equation (6) above as time goes to infinity is...

$$\lim_{t\to\infty} \text{Rate at time } t = \text{Rate at time infinity} \;\;\text{...because...}\;\; \lim_{t\to\infty} \text{Exp}\Bigg\{-\lambda\,t\Bigg\} = 0 \tag{7}$$

In the equations above we defined the variable λ to be the rate of mean reversion. To calibrate λ we will choose some future point in time (time = T) where the market rate is halfway between the rate at time zero and the rate at time infinity (i.e. the half life). The equation to calibrate λ is therefore... [1]

$$\text{Exp}\Bigg\{-\lambda\times T\Bigg\} = 0.50 \;\;\text{...such that...}\;\; \lambda = -\frac{\ln(0.50)}{T} \tag{8}$$

Using Equation (8) above and the data in Table 2 above the equation for the rate of mean reversion is...

$$\lambda = -\frac{\ln(0.50)}{3.00} = 0.2310 \tag{9}$$

Annualized Revenue

We defined notional value to be annualized operating revenue per Table 1 above. We will define the variable R_t to be annualized revenue at time t and the variable Γ_t to be the cumulative revenue rate over the time interval $[0,t]$.

The equation for annualized revenue at time t as a function of annualized reveue at time zero is...

$$R_t = R_0 \,\mathrm{Exp}\left\{\Gamma_t\right\} \tag{10}$$

We will define the variable μ_t to be the continuous-time revenue growth rate at time t. This rate is a function of the variable ω, which is the long-term sustainable rate, the variable Δ, which is the difference between the current unsustainable rate and the long-term sustainable rate, and the variable λ, which is the rate of mean reversion. The equation for the revenue growth rate at time t from the perspective of time zero is... [1]

$$\mu_t = \omega + \Delta \,\mathrm{Exp}\left\{ -\lambda\, t\right\} \text{ ...where... } \omega = \mu_\infty \text{ ...and... } \Delta = \mu_0 - \mu_\infty \tag{11}$$

We will define the variable Γ_t to be the cumulative revenue rate over the time interval $[0, t]$. Using Equation (11) above the equation for the cumulative revenue growth rate at time t from the perspective of time zero is... [1]

$$\Gamma_t = \int\limits_0^t \mu_s \,\delta s = \frac{\Delta}{\lambda} + \omega\, t - \frac{\Delta}{\lambda}\,\mathrm{Exp}\left\{ -\lambda\, t\right\} \tag{12}$$

Annualized revenue at time t is a function of annualized revenue at time zero and the cumulative revenue growth rate over the time interval $[0, t]$. Using Equations (10) and (12) above the equation for annualized revenue at time t from the perspective of time zero is...

$$R_t = R_0 \,\mathrm{Exp}\left\{\Gamma_t\right\} = R_0 \,\mathrm{Exp}\left\{\frac{\Delta}{\lambda} + \omega\, t - \frac{\Delta}{\lambda}\,\mathrm{Exp}\left\{ -\lambda\, t\right\}\right\} \tag{13}$$

The derivative of notional value Equation (13) above with respect to time is... [2]

$$\frac{\delta R_t}{\delta t} = R_0 \,\mathrm{Exp}\left\{\frac{\Delta}{\lambda} + \omega\, t - \frac{\Delta}{\lambda}\,\mathrm{Exp}\left\{ -\lambda\, t\right\}\right\}\left(\omega + \Delta \,\mathrm{Exp}\left\{ -\lambda\, t\right\}\right) \tag{14}$$

Note that because we are annualized revenue as an exponential the variables μ_0 and μ_∞ are continuous-time rates. Using the data in Table 2 above the equation for the continuous-time, short-term, unsustainable revenue growth rate is...

$$\mu_0 = \ln\left(1 + \text{annualized, discrete-time, short-term revenue growth rate}\right) = \ln\left(1 + 0.10\right) = 0.0953 \tag{15}$$

Using the data in Table 2 above the equation for the continuous-time, long-term, sustainable revenue growth rate is...

$$\mu_\infty = \ln\left(1 + \text{annualized, discrete-time, long-term revenue growth rate}\right) = \ln\left(1 + 0.04\right) = 0.0392 \tag{16}$$

Using Equations (10) to (16) above and the data in Table 2 above model parameters for our hypothetical problem are...

$$R_0 = 1,000,000 \text{ ...and... } \omega = 0.0392 \text{ ...and... } \Delta = 0.0953 - 0.0392 = 0.0561 \tag{17}$$

Assets

To calculate enterprise value we defined base value to be tangible operating assets per Table 1 above. We will define the variable A_t to be tangible operating assets at the end of time t and the variable ϕ to be the ratio of base value (assets) to notional value (revenue). Using Equations (2) and (10) above and the data in Table 2 above the equation for tangible operating assets at time t from the perspective of time zero is...

$$A_t = \phi\, R_t \text{ ...where... } A_0 = 1,250,000 \text{ ...and... } \phi = \frac{A_0}{R_0} = \frac{1,250,000}{1,000,000} = 1.25 \tag{18}$$

Using Equation (14) above the derivative of Equation (18) above with respect to time is...

$$\frac{\delta A_t}{\delta t} = \phi\,\frac{\delta R_t}{\delta t} = \phi\, R_0 \,\mathrm{Exp}\left\{\frac{\Delta}{\lambda} + \omega\, t - \frac{\Delta}{\lambda}\,\mathrm{Exp}\left\{ -\lambda\, t\right\}\right\}\left(\omega + \Delta \,\mathrm{Exp}\left\{ -\lambda\, t\right\}\right) \tag{19}$$

To calculate net cash flow we will need an equation for incremental investment, which we will define as the change in tangible operating assets. Using Equation (19) above the equation for incremental investment over the time interval $[t, t+\delta t]$ is...

$$\delta A_t = \phi\, R_0\, \mathrm{Exp}\left\{\frac{\Delta}{\lambda} + \omega\, t - \frac{\Delta}{\lambda}\,\mathrm{Exp}\left\{-\lambda\, t\right\}\right\}\left(\omega + \Delta\,\mathrm{Exp}\left\{-\lambda\, t\right\}\right)\delta t \tag{20}$$

We defined the variable θ_t to be the after-tax return on assets at time t. The rate of return is a function of the variable η, which is the long-term sustainable rate, the variable ψ, which is the difference between the current unsustainable rate and the long-term sustainable rate, and the variable λ, which is the rate of mean reversion. The equation for the rate of return at time t from the perspective of time zero is...

$$\theta_t = \eta + \psi\,\mathrm{Exp}\left\{-\lambda\, t\right\} \text{ ...where... } \eta = \theta_\infty \text{ ...and... } \psi = \theta_0 - \theta_\infty \tag{21}$$

To calculate an earnings growth rate we will need the derivative of the return on assets at time t. The derivative of Equation (21) above with respect to time is...

$$\frac{\delta\theta_t}{\delta t} = -\lambda\,\psi\,\mathrm{Exp}\left\{-\lambda\, t\right\} \text{ ...such that... } \delta\theta_t = -\lambda\,\psi\,\mathrm{Exp}\left\{-\lambda\, t\right\}\delta t \tag{22}$$

Using Equations (18) and (21) above the equation for annualized net income is...

$$\text{Annualized net income} = A_t\,\theta_t \tag{23}$$

Using Equations (19) and (22) above the derivative of Equation (23) above with respect to time via the chain rule is...

$$\frac{\delta \text{Annualized net income}}{\delta t} = \frac{\delta\, A_t\,\theta_t}{\delta t} = \frac{\delta A_t}{\delta t}\,\theta_t + \frac{\delta\theta_t}{\delta t}\,A_t \tag{24}$$

The earnings growth rate is defined as the ratio of the change in annualized net income to annualized income. We will define the variable E_t to be the continuous-time earnings growth rate at time t. Using Equations (23) and (24) above the equation for the earnings growth rate at time t from the perspective of time zero is...

$$E_t = \frac{\delta \text{ Annualized net income}}{\text{Annualized net income}} = \frac{\delta A_t\,\theta_t}{\delta t}\Bigg/A_t\,\theta_t = \frac{\delta A_t}{\delta t}\Bigg/A_t + \frac{\delta\theta_t}{\delta t}\Bigg/\theta_t \tag{25}$$

Using Equations (18) to (24) above and the data in Table 2 above model parameters for our hypothetical problem are...

$$\phi = 1.2500 \text{ ...and... } \eta = 0.1200 \text{ ...and... } \psi = 0.1800 - 0.1200 = 0.0600 \tag{26}$$

Interest-Bearing Debt

To calculate debt tax shield value we defined base value to be debt principal balance per Table 1 above. We will define the variable D_t to be debt principal balance at time t. Using Equations (2) and (10) above and the data in Table 2 above the equation for debt principal value at time t from the perspective of time zero is...

$$D_t = \epsilon\, R_t \text{ ...where... } D_0 = 300,000 \text{ ...and... } \epsilon = \frac{300,000}{1,000,000} = 0.3000 \tag{27}$$

Using Equation (14) above the derivative of Equation (27) above with respect to time is...

$$\frac{\delta D_t}{\delta t} = \epsilon\,\frac{\delta R_t}{\delta t} = \epsilon\, R_0\,\mathrm{Exp}\left\{\frac{\Delta}{\lambda} + \omega\, t - \frac{\Delta}{\lambda}\,\mathrm{Exp}\left\{-\lambda\, t\right\}\right\}\left(\omega + \Delta\,\mathrm{Exp}\left\{-\lambda\, t\right\}\right) \tag{28}$$

Using Equation (27) above and the data in Table 2 above model parameters for our hypothetical problem are...

$$\epsilon = 0.3000 \text{ ...and... } \iota = 0.0600 \text{ ...and... } \alpha = 0.2000 \tag{29}$$

Enterprise Value

We will define enterprise value as the discounted value of net operating cash flow over a given time interval. The equation for annualized net cash flow from operations at time t is...

$$\text{Annualized net cash flow} = \text{Annualized net income at time } t - \text{Annualized net investment at time } t \tag{30}$$

Using Equations (20) and (23) above we can rewrite Equation (30) above as...

$$\text{Annualized net cash flow} = A_t\,\theta_t - \delta A_t \tag{31}$$

We will define the variable $V_{m,n}$ to be the discounted value at time m of expected cash flow to be received over the time interval $[m,n]$ from the perspective of time zero. Using Equations (1) and (31) above the equation for enterprise value at time m from the perspective of time zero is...

$$\begin{aligned} V_{m,n} &= \int\limits_m^n \Big(A_t\,\theta_t - \delta A_t\Big)\text{Exp}\Big\{-\kappa\,(t-m)\Big\}\,\delta t \\ &= \text{Exp}\Big\{\kappa\,m\Big\}\int\limits_m^n \Big(A_t\,\theta_t - \delta A_t\Big)\text{Exp}\Big\{-\kappa\,t\Big\}\,\delta t \\ &= \text{Exp}\Big\{\kappa\,m\Big\}\bigg(\int\limits_m^n A_t\,\theta_t\,\text{Exp}\Big\{-\kappa\,t\Big\}\,\delta t - \int\limits_m^n \delta A_t\,\text{Exp}\Big\{-\kappa\,t\Big\}\,\delta t\bigg) \end{aligned} \tag{32}$$

Note that we can rewrite Equation (32) above as...

$$\text{if... } A = \int\limits_m^n A_t\,\theta_t\,\text{Exp}\Big\{-\kappa\,t\Big\}\,\delta t \text{ ...and... } B = \int\limits_m^n \delta A_t\,\text{Exp}\Big\{-\kappa\,t\Big\}\,\delta t \text{ ...then... } V_{m,n} = \text{Exp}\Big\{\kappa\,m\Big\}\Big(A - B\Big) \tag{33}$$

Using Appendix Equations (59) and (60) below we can rewrite enterprise value Equation (31) above as...

$$\begin{aligned} V_{m,n} = \text{Exp}\Big\{\kappa\,m\Big\}\,\phi\,N_0\bigg[&(\eta-\omega)\int\limits_m^n \text{Exp}\bigg\{\frac{\Delta}{\lambda} + (\omega-\kappa)\,t - \frac{\Delta}{\lambda}\,\text{Exp}\Big\{-\lambda\,t\Big\}\bigg\}\,\delta t \\ &+ (\psi-\Delta)\int\limits_m^n \text{Exp}\bigg\{\frac{\Delta}{\lambda} + (\omega-\kappa-\lambda)\,t - \frac{\Delta}{\lambda}\,\text{Exp}\Big\{-\lambda\,t\Big\}\bigg\}\,\delta t\bigg] \end{aligned} \tag{34}$$

We will integral one to be the following equation...

$$I_1 = \text{Exp}\Big\{\kappa\,m\Big\}\int\limits_m^n \text{Exp}\bigg\{\frac{\Delta}{\lambda} + c_1\,t - \frac{\Delta}{\lambda}\,\text{Exp}\Big\{-\lambda\,t\Big\}\bigg\}\,\delta t \text{ ...where... } c_1 = \omega - \kappa \tag{35}$$

We will integral two to be the following equation...

$$I_2 = \text{Exp}\Big\{\kappa\,m\Big\}\int\limits_m^n \text{Exp}\bigg\{\frac{\Delta}{\lambda} + c_2\,t - \frac{\Delta}{\lambda}\,\text{Exp}\Big\{-\lambda\,t\Big\}\bigg\}\,\delta t \text{ ...where... } c_1 = \omega - \kappa - \lambda \tag{36}$$

Using the integral definitions in Equations (35) and (36) above we can rewrite enterprise value Equation (34) above as...

$$V_{m,n} = \phi\,N_0\Big[(\eta-\omega)\,I_1 + (\psi-\Delta)\,I_2\Big] \tag{37}$$

Using Appendix Equation (56) below the solution to the integrals in Equation (37) above when the short-term notional value growth rate is less then the long-term notional value growth rate is...

$$\begin{aligned} &I = \text{Exp}\Big\{\kappa\,m\Big\}\,\text{Exp}\Big\{d\Big\}\,a^{\frac{c}{b}}\,b^{-1}\bigg[\Gamma\Big(\alpha, x_n\Big) - \Gamma\Big(\alpha, x_m\Big)\Big)\bigg] \text{ ...where...} \\ &a = \frac{\Delta}{\lambda} \text{ ...and... } b = \lambda \text{ ...and... } c_1 = \omega - \kappa \text{ ...and... } c_2 = \omega - \lambda - \kappa \text{ ...and... } d = \frac{\Delta}{\lambda} \text{ ...and...} \\ &\alpha = -\frac{c}{b} \text{ ...and... } x_m = a\,\text{Exp}\Big\{-b\,m\Big\} \text{ ...and... } x_n = a\,\text{Exp}\Big\{-b\,n\Big\} \text{ ...and... } \Delta > 0 \end{aligned} \tag{38}$$

Using Equation (38) above note the following...

$$\text{if... } m=0 \text{ ...then... } x_m = a\,\mathrm{Exp}\left\{-b\times 0\right\} = a \;\Bigg|\; \text{if... } n=\infty \text{ ...then... } x_n = a\,\mathrm{Exp}\left\{-b\times\infty\right\} = 0 \tag{39}$$

When the short-term notional value growth rate equals the long-term notional value growth rate then Δ in Equations (34), (35) and (36) above is equal to zero such that we can rewrite those integrals as...

$$I_1 = \int\limits_m^n \mathrm{Exp}\left\{(\omega-\kappa)\,t\right\}\delta t \text{ ...and... } I_2 = \int\limits_m^n \mathrm{Exp}\left\{(\omega-\kappa-\lambda)\,t\right\}\delta t \tag{40}$$

Using Appendix Equation (58) below the solutions to the integrals in Equation (40) above are...

$$I = \mathrm{Exp}\left\{\kappa\,m\right\}\int\limits_m^n \mathrm{Exp}\left\{x\,t\right\} = \mathrm{Exp}\left\{\kappa\,m\right\}\frac{1}{x}\left(\mathrm{Exp}\left\{x\,n\right\} - \mathrm{Exp}\left\{x\,m\right\}\right) \text{ ...where...}$$

$$x_1 = \omega-\kappa \text{ ...and... } x_2 = \omega-\lambda-\kappa \text{ ...and... } \Delta = 0 \tag{41}$$

Debt Tax Shield Value

We will define the variable ι to be the debt interest rate and the variable α to be the income tax rate. Since interest expense on debt is tax deductible this tax deduction has value (i.e. debt tax shield value). Using Equation (27) above the equation for annualized net cash flow applicable to the debt tax shield is...

$$\text{Annualized net cash flow at time } t = \iota\,D_t\,\alpha = \epsilon\,\iota\,\alpha\,R_t \tag{42}$$

We will define the variable $W_{m,n}$ to be debt tax shield value at time m. Debt tax shield value is defined as the discounted value at time m of expected tas savings applicable to the interest expense tax deduction to be received over the time interval $[m,n]$. Using Equations (1), (5) and (42) above the equation for debt tax shield value is...

$$W_{m,n} = \int\limits_m^n \epsilon\,\iota\,\alpha\,R_t\,\mathrm{Exp}\left\{-\kappa\,(t-m)\right\}\delta t \tag{43}$$

Using Equation (13) above we can rewrite Equation (43) above as...

$$\begin{aligned} S_{m,n} &= \mathrm{Exp}\left\{\kappa\,m\right\}\epsilon\,\iota\,\alpha\int\limits_m^n R_t\,\mathrm{Exp}\left\{-\kappa\,t\right\}\delta t \\ &= \mathrm{Exp}\left\{\kappa\,m\right\}\epsilon\,\iota\,\alpha\int\limits_m^n R_0\,\mathrm{Exp}\left\{\frac{\Delta}{\lambda}+\omega\,t-\frac{\Delta}{\lambda}\mathrm{Exp}\left\{-\lambda\,t\right\}\right\}\mathrm{Exp}\left\{-\kappa\,t\right\}\delta t \\ &= \mathrm{Exp}\left\{\kappa\,m\right\}\epsilon\,\iota\,\alpha\,R_0\int\limits_m^n \mathrm{Exp}\left\{\frac{\Delta}{\lambda}+(\omega-\kappa)\,t-\frac{\Delta}{\lambda}\mathrm{Exp}\left\{-\lambda\,t\right\}\right\}\delta t \end{aligned} \tag{44}$$

Using Appendix Equations (55) and (56) below we can rewrite the integral in Equation (44) above as...

$$I_1 = \mathrm{Exp}\left\{d\right\}a^{\frac{c}{b}}b^{-1}\left[\Gamma\left(\alpha,x_n\right)\right) - \Gamma\left(\alpha,x_m\right)\right)\right] \text{ ...where... } a=\frac{\Delta}{\lambda} \text{ ...and... } b=\lambda \text{ ...and... } c=\omega-\kappa$$

$$\text{...and... } d=\frac{\Delta}{\lambda} \text{ ...and... } \alpha=-\frac{c}{b} \text{ ...and... } x_m = a\,\mathrm{Exp}\left\{-b\,m\right\} \text{ ...and... } x_n = a\,\mathrm{Exp}\left\{-b\,n\right\} \tag{45}$$

Using Equation (45) above we can rewrite Equation (44) above as...

$$W_{m,n} = \mathrm{Exp}\left\{\kappa\,m\right\}\epsilon\,\iota\,\alpha\,R_0\,I_1 \tag{46}$$

The Answers To Our Hypothetical Problem

The table below presents ABC Company model parameters...

Table 3: Model Parameters

Parameter	Value	Reference	Parameter	Value	Reference
N_0	1,000,000	Equation (17)	a	0.7550	$a = \Delta \div \lambda$
Δ	0.0561	Equation (17)	b	0.2310	$b = \lambda$
η	0.1200	Equation (26)	c_1	-0.0645	$c_1 = \omega - \kappa$
κ	0.1133	Equation (5)	c_2	-0.2955	$c_2 = \omega - \kappa - \lambda$
λ	0.2310	Equation (9)	d	0.7550	$d = \Delta \div \lambda$
ω	0.0392	Equation (17)			
ϕ	1.2500	Equation (26)			
ψ	0.0600	Equation (26)			
ϵ	0.3000	Equation (29)			
ι	0.0600	Equation (29)			
α	0.2000	Equation (29)			

Question 1: What is company value at time zero given that cash flow is expected to be received over the time interval $[0, \infty]$?

Using Equation (38) above and the model parameter calculations in Table 3 above the equations for integral one and two are...

$$I_1 = 16.2536 \text{ ...and... } I_2 = 3.6463 \text{ ...and... } m = 0 \text{ ...and... } n = \infty \tag{47}$$

Using Equations (37) and (47) above and the model parameters in Table 3 above the equation for enterprise value is...

$$V_{0,\infty} = 1.25 \times 1,000,000 \times \Bigg[(0.1200 - 0.0392) \times 16.2536 + (0.0600 - 0.0561) \times 3.6463\Bigg] = 1,659,400 \tag{48}$$

Using Equations (46) and (47) above and the model parameters in Table 3 above the equation for debt tax shield value is...

$$W_{0,\infty} = \text{Exp}\Bigg\{0.1133 \times 0\Bigg\} \times 0.30 \times 0.06 \times 0.20 \times 1,000,000 \times 16.2536 = 58,500 \tag{49}$$

Using Equation (2) above the answer to the question is...

$$\text{Company value} = 1,659,400 + 58,500 = 1,717,900 \tag{50}$$

Question 2: What is company value at the end of year 5 given that cash flow is expected to cease after year 15?

Using Equation (38) above and the model parameter calculations in Table 3 above the equations for integral one and two are...

$$I_1 = 10.5918 \text{ ...and... } I_2 = 1.4577 \text{ ...and... } m = 5 \text{ ...and... } n = 15 \tag{51}$$

Using Equations (37) and (47) above and the model parameters in Table 3 above the equation for enterprise value is...

$$V_{5,15} = 1.25 \times 1,000,000 \times \Bigg[(0.1200 - 0.0392) \times 10.5918 + (0.0600 - 0.0561) \times 1.4577\Bigg] = 1,076,900 \tag{52}$$

Using Equations (46) and (47) above and the model parameters in Table 3 above the equation for debt tax shield value is...

$$W_{5,15} = \text{Exp}\Bigg\{0.1133 \times 5.00\Bigg\} \times 0.30 \times 0.06 \times 0.20 \times 1,000,000 \times 10.5918 = 58,500 \tag{53}$$

Using Equation (2) above the answer to the question is...

$$\text{Company value} = 1,076,900 + 58,500 = 1,135,400 \tag{54}$$

References

[1] Gary Schurman, *Return Models - The Stochastic, Mean-Reverting Short Rate,* November, 2017

[2] Gary Schurman, *Incomplete Gamma Function - A Mean-Reverting Return Model,* November, 2017

Appendix

A. Note that we can rewrite the integrals in Equation (**??**) above to be in the following form...

$$I=\int\limits_m^n \mathrm{Exp}\bigg\{d+c\,t-a\,\mathrm{Exp}\bigg\{-b\,t\bigg\}\bigg\}\ \text{...where...}\ a=\frac{\Delta}{\lambda}\ \text{...and...}\ b=\lambda\ \text{...and...}$$

$$c_1=\omega-\kappa\ \text{...and...}\ c_2=\omega-\kappa-\lambda\ \text{...and...}\ d=\frac{\Delta}{\lambda} \tag{55}$$

The solution to the integral in Equation (55) above is... [2]

$$I=\mathrm{Exp}\bigg\{d\bigg\}\,a^{\frac{c}{b}}\,b^{-1}\bigg[\Gamma\bigg(\alpha,x_n\bigg)\bigg)-\Gamma\bigg(\alpha,x_m\bigg)\bigg)\bigg]\ \text{...where...}\ \alpha=-\frac{c}{b}\ \text{...and...}$$

$$x_m=a\,\mathrm{Exp}\bigg\{-b\,m\bigg\}\ \text{...and...}\ x_n=a\,\mathrm{Exp}\bigg\{-b\,n\bigg\} \tag{56}$$

Note that the solution to the upper incomplete gamma function $\Gamma(\alpha,x)$ using standard Excel functions is... [**?**]

$$\Gamma(\alpha,x)=\mathrm{EXP(GAMMALN}(\alpha))\times(1-\mathrm{GAMMA.DIST(x,}\ \alpha,\ 1,\ \mathrm{true})) \tag{57}$$

B. When $\Delta=0$ the solution to the integral in Appendix Equation (55) above becomes...

$$I=\int\limits_m^n \mathrm{Exp}\bigg\{c\,t\bigg\}=\frac{1}{c}\,\mathrm{Exp}\bigg\{c\,t\bigg\}\bigg[_m^n=\frac{1}{c}\bigg(\mathrm{Exp}\bigg\{c\,n\bigg\}-\mathrm{Exp}\bigg\{c\,m\bigg\}\bigg)$$

$$\text{...where...}\ c_1=\omega-\kappa\ \text{...and...}\ c_2=\omega-\kappa-\lambda \tag{58}$$

C. Using Equations (12), (18) and (21) above we can rewrite the first integral in Equations (33) above as...

$$\begin{aligned}A&=\int\limits_m^n \phi\,N_0\,\mathrm{Exp}\bigg\{\frac{\Delta}{\lambda}+\omega\,t-\frac{\Delta}{\lambda}\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\bigg\}\bigg(\eta+\psi\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\bigg)\,\mathrm{Exp}\bigg\{-\kappa\,t\bigg\}\,\delta t\\&=\phi\,N_0\int\limits_m^n \mathrm{Exp}\bigg\{\frac{\Delta}{\lambda}+(\omega-\kappa)\,t-\frac{\Delta}{\lambda}\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\bigg\}\bigg(\eta+\psi\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\bigg)\,\delta t\\&=\phi\,N_0\bigg[\eta\int\limits_m^n \mathrm{Exp}\bigg\{\frac{\Delta}{\lambda}+(\omega-\kappa)\,t-\frac{\Delta}{\lambda}\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\bigg\}\,\delta t+\psi\int\limits_m^n \mathrm{Exp}\bigg\{\frac{\Delta}{\lambda}+(\omega-\kappa-\lambda)\,t-\frac{\Delta}{\lambda}\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\bigg\}\,\delta t\bigg]\end{aligned} \tag{59}$$

D. Using Equations (12), (18) and (20) above we can rewrite the second integral in Equation (33) above as...

$$\begin{aligned}B&=\int\limits_m^n \phi\,N_0\,\mathrm{Exp}\bigg\{\frac{\Delta}{\lambda}+\omega\,t-\frac{\Delta}{\lambda}\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\bigg\}\bigg(\omega+\Delta\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\bigg)\,\mathrm{Exp}\bigg\{-\kappa\,t\bigg\}\,\delta t\\&=\phi\,N_0\int\limits_m^n \mathrm{Exp}\bigg\{\frac{\Delta}{\lambda}+(\omega-\kappa)\,t-\frac{\Delta}{\lambda}\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\bigg\}\bigg(\omega+\Delta\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\bigg)\,\delta t\\&=\phi\,N_0\bigg[\omega\int\limits_m^n \mathrm{Exp}\bigg\{\frac{\Delta}{\lambda}+(\omega-\kappa)\,t-\frac{\Delta}{\lambda}\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\bigg\}\,\delta t+\Delta\int\limits_m^n \mathrm{Exp}\bigg\{\frac{\Delta}{\lambda}+(\omega-\kappa-\lambda)\,t-\frac{\Delta}{\lambda}\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\bigg\}\,\delta t\bigg]\end{aligned} \tag{60}$$